where the higher order terms in the asymptotic expansion are identically zero as the zero-order terms give an exact representation of the exoatmospheric Keplerian motion. Equations (11) may now be integrated to give the exact outer solution as

$$u_0 = u_{00} + \frac{2}{1+h} \tag{12a}$$

$$w_0 = \frac{w_{00}}{\sqrt{u_{00}(1+h)^2 + 2(1+h)}} \tag{12b}$$

$$m_0 = m_{00} \tag{12c}$$

Clearly the projectile mass is constant during exoatmospheric flight as expected.

To now determine the integration constants of the outer solution, u_{00}, w_{00}, and m_{00}, an asymptotic matching procedure must now be used. Matching the inner and outer solutions, the unknown outer-solution integration constants are obtained in terms of the inner-solution constants. The inner constants may then be related to the projectile initial conditions.

Composite Solution

Matching the inner and outer solution constants,[3,4] it is found that the uniformly valid composite solution representing ablative projectile motion may now be written to lowest order as

$$u(h) = \eta^{-1} Ei^{-1}\left\{ Ei\{\eta \tilde{u}_{00}\} + \frac{\tilde{m}_{00}^{\alpha-1} C_D}{\sqrt{1-\tilde{w}_{00}^2}} \times \tilde{\rho}_0\{e^{-h/\varepsilon} - e^{-h_0/\varepsilon}\}\right\} - \frac{2h}{1+h} \tag{13a}$$

$$w(h) = \tilde{w}_{00}\frac{\sqrt{u_{00}+2}}{\sqrt{u_{00}(1+h)^2 + 2(1+h)}} \tag{13b}$$

$$\bar{m}(h) = \exp\left\{ -\frac{\eta}{1-\alpha}[\tilde{u}_{00} - \tilde{u}_0(h)]\right\} \tag{13c}$$

where the inner velocity solution $\tilde{u}_0$ is obtained from Eq. (9a). These solutions are uniformly valid in the entire domain of projectile motion and represent an ablating ballistic projectile trajectory to lowest order in ε.

Implementation

The composite solutions defined by Eqs. (13) will now be used to compute the trajectory of a hypervelocity ablating projectile. As a numerical example, a projectile fired from sea level with a launch tube elevation of 45 deg will be considered. For ease of illustration the projectile parameters have been chosen ($\eta = 2.56$ and $C_D = 0.1$) to induce rapid ablation of the projectile during its atomspheric pass. The velocity profile shown in Fig. 2 shows rapid drag loss during ascent through the lower atmosphere. On exit from the atmosphere the projectile follows a ballistic Keplerian arc, as expected. Similarly, the projectile mass falls rapidly as material is thermally ablated in the lower atmosphere, Fig. 2. On exit from the atmosphere the projectile has a constant residual mass.

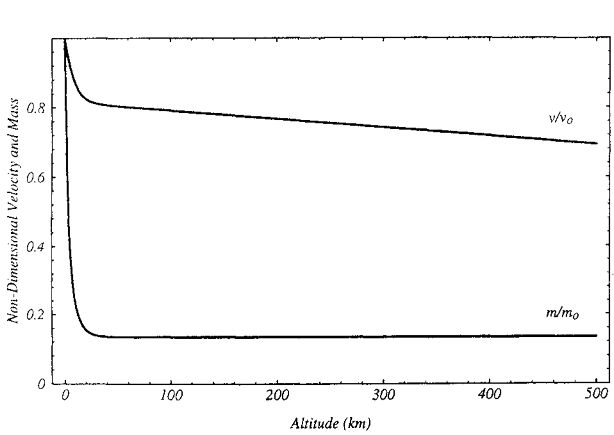

Fig. 2 Projectile velocity and mass profile.

References

[1]Kaloupis, P., and Bruckner, A. P., "The Ram Accelerator: A Chemically Driven Mass Launcher," AIAA Paper 88-2968, July 1988.

[2]Bogdanoff, D. W., "Ram Accelerator Direct Space Launch System: New Concepts," *Journal of Propulsion and Power*, Vol. 8, No. 2, 1992, pp. 481–490.

[3]McInnes, C. R., "Aero-Assisted Trajectories for Direct Launch Systems," *Acta Astronautica*, Vol. 32, No. 6, 1994, pp. 411–417.

[4]O'Neill, C. F., and McInnes, C. R., "Matched Asymptotic Solutions for a Hypervelocity Atmospheric Entry Vehicle," 44th International Astronautical Federation Congress, IAF-93-A.1.3, Graz, Austria, Oct. 1993.

[5]Bronshten, V. A., *Physics of Meteoric Phenomena*, Reidel, Dordrecht, The Netherlands, 1981.

[6]Kevorkian, J., and Cole, J. D., *Perturbation Methods in Applied Mathematics*, Applied Mathematical Sciences, Vol. 34, Springer–Verlag, New York, 1981, pp. 20–28.

[7]Vinh, N. X., Buseman, A., and Culp, R. D., *Hypersonic and Planetary Entry Flight Mechanics*, Univ. of Michigan Press, Ann Arbor, MI, 1980.

Approximate Solution to Lawden's Problem

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Introduction

LAWDEN[1] first solved the problem of optimally transferring between two coplanar ellipses whose size and shape are identical but whose lines of apsides are not aligned. Marchal[2] showed that a two-impulse transfer is optimum when the eccentricity is less than 0.535. Lawden's solution provided the true anomaly of the impulse point and the impulse direction of the two symmetric impulses. Unfortunately, his solution required the iterative satisfaction of six simultaneous equations. These equations do not readily provide insight into the effect of eccentricity, say, on the impulse magnitude. Other studies have attempted to overcome this drawback by finding approximate solutions using various simplifying assumptions. For example, Bender[3] and Kuzmak[4] both obtained simpler solutions by assuming that the two impulses are separated by 180 deg. Karrenberg[5] solved for the impulse location and magnitude by assuming that the impulse direction is tangential. In addition, the solutions of both Kuzmak and Karrenberg are restricted to low-eccentricity orbits.

Many three-axis stabilized spacecraft employ a vehicle-fixed coordinate frame where one axis points toward Earth's center. A second axis, orthogonal to the first and lying in the orbit plane, usually defines the thruster pointing direction. From a practical viewpoint, then, it would be useful to have a solution to Lawden's problem where the impulse direction is circumferential and where the results are obtained explicitly rather than iteratively.

Analysis

Given an ellipse of semimajor axis a and eccentricity e, the problem is to rotate the line of apsides through an angle $\Delta\omega$. Following Lawden, assume that the transfer is accomplished by two symmetrical, equal-magnitude impulses, each of which rotates the line of

Received July 28, 1994; revision received Dec. 6, 1994; accepted for publication Dec. 19, 1994.

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apsides by $\Delta\omega/2$. However, the direction of these impulses is assumed to be circumferential. At the first impulse point, the energy equation gives

$$V_t^2 = V^2 + V_c^2(1 - a/a_t) \tag{1}$$

where V is the velocity, the subscript t refers to the transfer orbit, a lack of subscript refers to the initial orbit, and the circular velocity V_c is defined by

$$V_c = \sqrt{\mu/a}$$

where μ is the gravitational constant.

Solving the velocity triangle at the first impulse point gives

$$V_t^2 = V^2 + (\Delta V)^2 + 2V\Delta V\cos\gamma \tag{2}$$

where γ is the flight-path angle. The angular momentum equation gives

$$V\cos\gamma = V_c(1 + e\cos f)\big/\sqrt{1-e^2} \tag{3}$$

where f is the true anomaly. Equations (1) and (2) can be used to eliminate V_t, yielding an equation that is quadratic in ΔV but that also contains the transfer orbit semimajor axis. From the constancy of the radius at the first impulse point, one obtains

$$\frac{a}{a_t} = \frac{\left(1 - e_t^2\right)(1 + e\cos f)}{(1 - e^2)(1 + e_t\cos f_t)} \tag{4}$$

where $f_t = f - \Delta\omega/2$. Similarly, the constancy of the radial velocity at the first impulse point gives

$$e_t^2\sin^2 f_t(1 + e\cos f) = e^2\sin^2 f(1 + e_t\cos f_t) \tag{5}$$

Assuming that the initial orbit eccentricity is small, one can solve Eq. (5) for the transfer orbit eccentricity in terms of e. Dropping higher order terms of e yields

$$e_t \approx \frac{e\sin f}{\sin f_t\sqrt{(1 + e\cos f)}} + \frac{\frac{1}{2}e^2\sin^2 f\cos f_t}{\sin^2 f_t(1 + e\cos f)} \tag{6}$$

Equations (4) and (6) can be used to eliminate a/a_t and e_t from the quadratic equation for ΔV, yielding

$$0 \approx \left(\frac{\Delta V}{V_c}\right)^2 + 2\frac{(1 + e\cos f)}{\sqrt{(1 - e^2)}}\frac{\Delta V}{V_c} + C$$

where C is of order e and is given approximately by

$$C \approx -\frac{e\sin(\Delta\omega/2)\left[1 + e\cos f + \frac{1}{2}e\sin f/\tan f_t\right]}{\left[(1 - e^2)\sin f_t\right]}$$

Again dropping higher order terms of e, one obtains an explicit equation for the total ΔV for the two-part maneuver:

$$\frac{\Delta V_{\text{total}}}{V_c} \approx \frac{e\sin(\Delta\omega/2)F}{\sqrt{(1 - e^2)}\sin(f - \Delta\omega/2)} \tag{7}$$

where F is defined by

$$F = 1 + \frac{1}{2}\frac{e\sin f}{\tan(f - \Delta\omega/2)} - \frac{\frac{1}{4}e\sin(\Delta\omega/2)}{\sin(f - \Delta\omega/2)} \tag{8}$$

Equation (7) shows that the normalized total impulse is not only independent of the semimajor axis but also varies linearly with eccentricity and with $\Delta\omega$ for small eccentricities and small rotations of the line of apsides.

The optimum true anomaly for the first impulse is found by differentiating Eq. (7) with respect to f and setting the resultant expression equal to zero, giving

$$f_{\text{opt}} \approx 90 + \frac{\Delta\omega}{2} + \frac{\frac{1}{2}e\cos(\Delta\omega/2)}{1 + \frac{1}{2}e\sin(\Delta\omega/2)} \tag{9}$$

Equation (7) is similar to the expression obtained by Karrenberg for the total ΔV. For very small e, Eq. (9) reduces to Karrenberg's optimum true anomaly as well.

Results

The total normalized ΔV is shown in Fig. 1 for all possible changes in the line of apsides and for several orbit eccentricities. For comparison, the Lawden optimum and the tangential impulse results of Karrenberg are included. It is seen that Eq. (7) provides a more accurate representation of the Lawden solution than the tangential impulse results. For many applications, the penalty for using circumferential impulses rather than the Lawden optimum solution is vanishingly small.

The optimum location of the first impulse for the case where $e = 0.3$ is shown in Fig. 2. Equation (9) gives a solution close to the Lawden optimum, whereas the tangential impulse location is less accurate. Similar results are obtained for other eccentricities. From symmetry arguments, the true anomaly of the second impulse

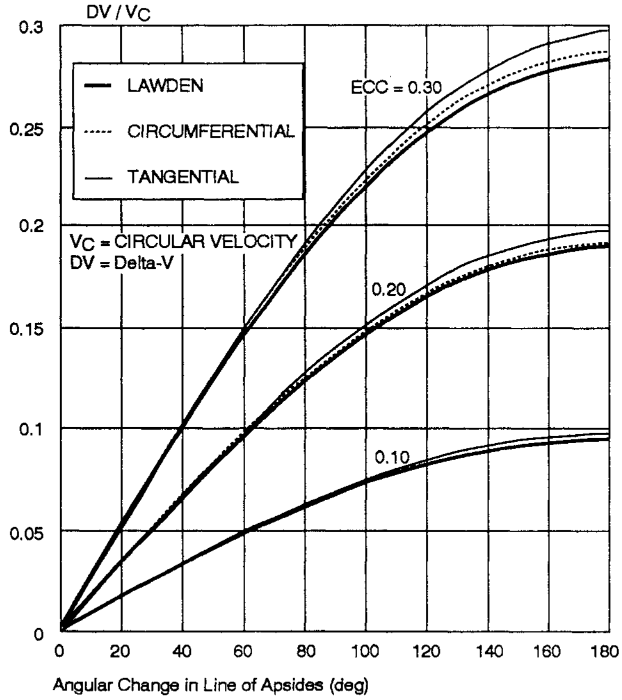

Fig. 1 Normalized ΔV.

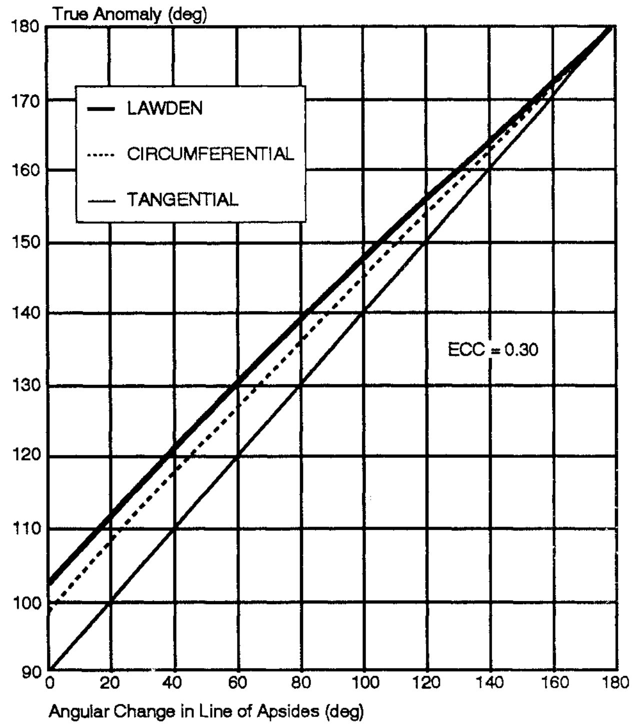

Fig. 2 Optimum impulse location.

is equal to $(360 - f_{\mathrm{opt}})$. Clearly, the two impulses generally are not separated by 180 deg. Hence, one would expect that the solutions of Bender and Kuzmak for $\Delta V_{\mathrm{total}}$ would be less accurate as well.

Conclusions

The use of the two circumferential impulses to rotate the line of the apsides without changing the semimajor axis or eccentricity produces results that are in very good agreement with Lawden's optimal solution. Insight into the effect of eccentricity or line of apsides change is obtained from the simple approximate expressions for the magnitude and the location of the impulses.

References

[1]Lawden, D. F., "Impulsive Transfer Between Elliptical Orbits," *Optimization Techniques*, Academic, New York, 1962, pp. 323–351.

[2]Marchal, C., "Transferts Optimaux Entre Orbites Elliptiques (Duree Indifferente)," presented at XVI International Astronautical Federation Congress, Athens, 1965.

[3]Bender, D. F., "Optimum Coplanar Two-Impulse Transfers between Elliptical Orbits," *Aerospace Engineering*, Vol. 21, Oct. 1962.

[4]Kuzmak, G. E., "Linearized Theory of Optimal Multi-Impulse Plane Flights," *Cosmic Research*, Vol. 3, No. 1, 1965.

[5]Karrenberg, H. K., "Cotangential Transfer between Equal Ellipses Differing Only in Argument of Perigee," Aerospace Corp., ATM 81 (6463-05)-10, El Segundo, CA, Sept. 30, 1981.